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A Terminal Guidance Theory Using Dynamic Programing Formulation

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The dynamic programing condition for minimum error (cost function) for a general η dimensional dynamic process is extended to optimization problems subject to integral-type constraints. The generalized dynamic programing formulation is then applied to the implicit terminal guidance problems. An optimal guidance law in the neighborhood of a nominal (optimal or not) trajectory is developed. This guidance law is optimal in the sense of guiding a vehicle from an arbitrary starting point to an arbitrary fixed-end point along a neighborhood trajectory with minimum control energy consumption. It is not approximate optimization in the same sense of the nominal trajectory if an optimal trajectory in some sense is used as nominal. The resulting guidance law is a closed-loop feedback control in terms of state variables and Lagrange multipliers. The equations for computing the Lagrange multipliers to satisfy the terminal constraints are derived. Thus, a control law in terms of only the terminal constraints, initial conditions, and state variables is obtained. This control law also can be applied to optimization and control of nonlinear systems and solving two-point boundary problems.

I. Introduction

THE problem of terminal guidance has been studied previously by Kelley,[1,2] Breakwell, Bryson, and Spener.[3,4] In all these references, an optimized nominal trajectory resulting from variational calculus, which minimizes (or maximizes) a function of certain terminal quantities while satisfying specified initial and terminal conditions, was used as a reference trajectory. The properties of the optimal nominal trajectory were employed to develop a feedback guidance law that is approximately optimal in the same sense as the nominal trajectory. The terminal quantities being controlled or optimized vary with initial conditions and control scheme used. The guidance schemes developed in these references fail in cases in which the nominal trajectory is not optimal and all the terminal quantities are specified. Furthermore, to compute the optimal nominal trajectory is a difficult two-point boundary-value problem. To avoid the difficulty of obtaining an optimal trajectory and to solve the guidance problem that all the terminal quantities are specified, a terminal guidance theory with a different philosophy is developed in this paper.

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Here, implicit guidance philosophy is used and any obtainable trajectory (not necessarily optimal) between two end points and the associated control policy are used as nominal trajectory and reference control. The guidance problem treated here is to guide a vehicle from the initial point region to the terminal point region along the neighborhood of the nominal trajectory with minimum control effort. All the terminal values of the state variables are specified (a fixed-end point problem), and a cost function related to the control effort is set up to be minimized. The optimal control law is derived using dynamic programing formulation to provide zero terminal errors by using the desired terminal conditions as terminal constraints in the optimization process.

The design of optimum control systems using dynamic programing formulation has been treated fully in the literature. The dynamic programing condition for minimum error for a general η dimensional dynamic process is a partial differential equation of the minimum error (cost) function,[5] and leads to free-point terminal-boundary conditions. In Ref. 5, the minimum error function is assumed to be a quadratic form of the state variables of the system, and thus results in a feedback optimal control law that is linear in the state variables.

Optimization problems subject to an integral constraint for a first-order dynamic process also were treated in Ref. 5. This can introduce fixed-point terminal conditions into the dynamic programing formulation of the condition for minimum error. In this paper, the dynamic programing condition for minimum error for a general η dimensional dynamic process subject to integral-type constraints is derived. It is shown that the minimization subject to integral constraints can be treated simply by an extension of dynamic programing using Lagrange multipliers. The generalized dynamic programing formulation can be applied to the optimal terminal control of linear and nonlinear systems. For linear system dynamics, and quadratic error measurement, the minimum error function is in quadratic in the state variables,[6] and the resulting control law is a closed-loop feedback control in the terms of state variables and the Lagrange multipliers. The Lagrange multipliers must be adjusted to satisfy the terminal conditions. Conventionally, the Lagrange multipliers are determined after the state differential equations are solved, if an analytical solution can be obtained. Otherwise, a trial and error procedure is used to the Lagrange multipliers in the computation of the control law to satisfy the terminal condition, and the choice of Lagrange multipliers to satisfy the terminal constraints is a two-point-boundary-value problem.

In this paper, it is found that for linear system, the Lagrange multipliers that satisfy the terminal conditions can be solved in terms of terminal and initial conditions. Thus, the Lagrange multipliers are eliminated from the control law to yield a guidance law in terms of initial conditions, terminal conditions, and state variables. This property is employed to develop an optimal terminal guidance law with minimum control energy consumption for aerospace vehicles using implicit guidance philosophy. The formulas for computing the required Lagrange multipliers in terms of initial conditions and terminal constraints are derived, and the optimal guidance law suitable for onboard guidance computer is developed. Thus, the two-point-boundary-value problem of adjusting the Lagrange multipliers to satisfy the terminal constraints in the computation of the control law is avoided. This control scheme also can be applied to optimization and control of nonlinear systems and solving the two-point-boundary-value problems.

II. Generalized Dynamic Programing Conditions for Minimum Error with Integral Constraints

Let an η dimensional dynamic process be described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{m}(t),t] \tag{1}$$

$$\mathbf{g}(t) = \mathbf{g}[\mathbf{x}(t),t] \tag{2}$$

$$\mathbf{m}(t)\,\mathcal{E}\,\mathfrak{M}(t)$$

Where the vectors $\mathbf{x}(t)$, $\mathbf{g}(t)$, and $\mathbf{m}(t)$ are state, output, and input vectors, respectively, of the system. The notation $(\dot{\ }) = d(\ )/dt$, and $\mathbf{m}(t)\mathcal{E}\mathfrak{M}(t)$ means the vector $\mathbf{m}(t)$ lies within or on the boundary of a closed region $\mathfrak{M}(t)$ of the vector space. These vectors have n, Q, and M components such that

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ g_Q(t) \end{bmatrix}, m(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ m_M(t) \end{bmatrix} \tag{3}$$

The error index is taken to be

$$e(t) = \int_t^{t_f} h[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma]d\sigma \tag{4}$$

with the constraints

$$y_j(t) = \int_t^{t_f} I_j[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma]d\sigma,\ j = 1,2,\ldots,L \tag{5}$$

If the constraints of Eq. (5) are not considered, the minimum error function was given in Ref. 5 as

$$E[\mathbf{x}(\mu),\mu] = \underset{[\mu,t_f]}{\mathbf{m}(\sigma) \overset{\text{Min}}{\mathcal{E}} \mathfrak{M}(\sigma)} \int_\mu^{t_f} h[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma]d\sigma \tag{6}$$

where

$$E[\mathbf{x}(t_f),t_f] = 0 \tag{7}$$

The condition for minimum error was given as

$$\mathbf{m}(u) \overset{\text{Min}}{\mathcal{E}} \mathfrak{M}(u)\{h\{[\mathbf{g}(\mu),\mathbf{m}(\mu),(\mu],\mu\} + dE[\mathbf{x}(\mu),\mu]/d\mu\} = 0 \tag{8}$$

If the total time derivative of the minimum error is expanded in terms of partial derivatives and Eqs. (1) and (2) are substituted into the condition for minimum error, the condition for minimum error becomes

$$\mathbf{m}(u) \overset{\text{Min}}{\mathcal{E}} \mathfrak{M}(\mu) \left(h\{\mathbf{g}[\mathbf{x}(\mu),\mu],\mathbf{m}(\mu),\mu\} + \sum_{i=1}^{n} f_i[\mathbf{x}(\mu),\mathbf{m}(\mu),u] \times \frac{\partial E[\mathbf{x}(\mu),\mu]}{\partial x_i(\mu)} \right) = - \frac{\partial E[\mathbf{x}(\mu),\mu]}{\partial \mu} \tag{9}$$

In the case of minimization without constraints, the minimum error function as given by Eq. (6) is a function of the initial condition of the state variables and initial time only. However, when the system is subject to a constraint vector $\mathbf{y}(t)$ whose components are given by Eq. (5), the minimum error function is also a function of the constraint vector $\mathbf{y}(t)$. Therefore, the minimum error function is defined as

$$E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu] = \underset{[\mu,t_f]}{\mathbf{m}(\sigma) \overset{\text{Min}}{\mathcal{E}} \mathfrak{M}(\sigma)} \int_\mu^{t_f} h[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma]d\sigma \tag{10}$$

where

$$E[\mathbf{x}(t_f),\mathbf{y}(t_f),t_f] = 0 \tag{11}$$

Then, it can be shown that the condition for minimum error is given as

$$\mathbf{m}(\mu) \overset{\text{Min}}{\mathcal{E}} \mathfrak{M}(\mu) \left\{ h[\mathbf{g}(\mu),\mathbf{m}(\mu),\mu] + \frac{dE[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{d\mu} \right\} = 0 \tag{12}$$

The total time derivative of the minimum error function is expanded in terms of partial derivatives as

$$\frac{dE[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{d\mu} = \frac{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial \mu} + \sum_{i=1}^{n} \dot{x}_i(\mu) \times \frac{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial x_i(\mu)} + \sum_{j=1}^{L} \dot{y}_j(\mu) \frac{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial y_j(\mu)} \tag{13}$$

The time derivatives $\dot{y}_j(\mu)$ are found by differentiating Eq. (5). Under the condition that $\mathbf{m}(\mu)$ assume the minimizing values $\mathbf{m}^*(\mu)$, the values of these derivatives are

$$\dot{y}_j^*(\mu) = -I_j[\mathbf{g}(\sigma),\mathbf{m}^*(\sigma),\sigma],\ j = 1,2,\ldots,L \tag{14}$$

Also, Eq. (12) reduces to

$$h[\mathbf{g}(\mu),\mathbf{m}^*(\mu),\mu] + \frac{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial \mu} + \sum_{i=1}^{n} f_i[\mathbf{x}(\mu),\mathbf{m}^*(\mu),\mu] \times \frac{\partial E[\mathbf{x}(\mu),y(\mu),\mu]}{\partial x_i(\mu)} + \sum_{j=1}^{L} \dot{y}_j^*(\mu) \frac{\partial E[\mathbf{x}(\mu),y(\mu),\mu]}{\partial y_j(\mu)} = 0 \tag{15}$$

When the condition for minimum error is satisfied, the partial derivatives of Eq. (15) with respect to $y_k(\mu)$ are

$$\frac{\partial^2 E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial\mu\partial y_k(\mu)} + \sum_{j=1}^{n} f_i[\mathbf{x}(\mu).\mathbf{m}^*(\mu)\ \mu]\,\frac{\partial^2 E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial x_i(\mu)\partial y_k(\mu)} + \sum_{j=1}^{L} \dot{y}_j{}^*\,\frac{\partial^2 E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial y_j(\mu)\partial y_k(\mu)} = 0, \qquad k = 1,\ldots,L \quad (16)$$

which are written in the alternative form

$$d/d\mu\{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]/\partial y_k\} = 0,\ k = 1,\ldots,L \quad (17)$$

Therefore, along the optimum trajectory as defined by $\mathbf{m}^*(\mu)$, the partial derivatives of the minimum error function with respect to the constraint variables are constants. If Eq. (17) is integrated with respect to μ, then

$$\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]/\partial y_k = -2\lambda_k,\ k = 1,\ldots,L \quad (18)$$

where the constants of integration are defined as $-2\lambda_k$.

Substituting Eqs. (14) and (18) into Eq. (15) yields

$$h[\mathbf{g}(\mu),\mathbf{m}^*(\mu),\mu] + 2\sum_{j=1}^{L}\lambda_j I_j[\mathbf{g}(\mu),\mathbf{m}^*(\mu),\mu] + \sum_{i=1}^{n} f_i[\mathbf{x}(\mu),\mathbf{m}^*(\mu),\mu]\,\frac{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial x_i(\mu)} = -\frac{\partial E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu]}{\partial\mu} \quad (19)$$

It is seen that if the error measure of the constraint problem is replaced by a cost function of the form as

$$e_c(\mu) = \int_{\mu}^{t_f}\left\{h[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma] + 2\sum_{j=1}^{L}\lambda_j I_j[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma]\right\}d\sigma \quad (20)$$

the minimum error function of the constraint problem is defined as

$$E_c[\mathbf{x}(\mu),\mu] = E[\mathbf{x}(\mu),\mathbf{y}(\mu),\mu] = \underset{[\mu,t_f]}{\mathbf{m}(\sigma)\ \overset{\text{Min}}{\boldsymbol{\varepsilon}}\ \mathfrak{M}}(\sigma) \times \int_{\mu}^{t_f}\left\{h[\mathbf{g}(\sigma),(\sigma),\mathbf{m}\sigma] + 2\sum_{j=1}^{L}\lambda_j I_j[\mathbf{g}(\sigma),\mathbf{m}(\sigma),\sigma]\right\}d\sigma \quad (21)$$

The condition for minimum error corresponding to this definition of the minimum error function results in Eq. (19). Therefore, minimization subject to integral constraints of the type given in Eq. (5) can be treated simply by an extension of dynamic programing condition using Lagrange multipliers. The condition for minimum error of the constraint problem becomes

$$\mathbf{m}(\mu)\ \overset{\text{Min}}{\boldsymbol{\varepsilon}}\ \mathfrak{M}(\mu)\left(h\,\{\mathbf{g}[\mathbf{x}(\mu),\mu],\mathbf{m}(\mu),\mu\} + 2\sum_{j=1}^{L}\lambda_j I_j\{\mathbf{g}[\mathbf{x}(\mu),\mu],\mathbf{m}(\mu),\mu\} + \sum_{i=1}^{n} f_i[\mathbf{x}(\mu),\mathbf{m}(\mu),\mu]\,\frac{\partial E_c[\mathbf{x}(\mu),\mu]}{\partial x_i(\mu)}\right) = -\frac{\partial E_c[\mathbf{x}(\mu),\mu]}{\partial\mu} \quad (22)$$

The type of problem discussed here is called an isoperimetric problem. One significant use of the isoperimetric problem is the introduction of fixed-point terminal-boundary conditions into the dynamic programing formulation of the condition for minimum error. This is done by taking I_j of the constraint Eq. (5) as

$$I_j = \dot{x}_j(\sigma) = f_j[\mathbf{x}(\sigma),\mathbf{m}(\sigma),\sigma],\ j = 1,\ldots,L \quad (23)$$

so that the constraint vector appearing as

$$y(\mu) = \int_{\mu}^{t_f}\dot{\mathbf{x}}(\sigma)d\sigma = \mathbf{x}(t_f) - \mathbf{x}(\mu) \quad (24)$$

This constraint condition is used to apply the dynamic programing formulation to solve the optimal terminal guidance problems in this paper.

In this formulation, the problem lies on the solution of the partial differential Eq. (22) for the minimum error function E_c and the adjustment of the Lagrange multipliers λ_j to satisfy the terminal conditions. If the system dynamics is linear and the error measurement is quadratic, the minimum error function E_c is in quadratic form of the state variables $\mathbf{x}$.[6]

III. Optimal Terminal Guidance of Aerospace Vehicles

The system to be controlled is described by a set of first-order nonlinear differential equations

$$\dot{\boldsymbol{\theta}} = \mathbf{F}(\boldsymbol{\theta},\mathbf{u},t) \quad (25)$$

where $\boldsymbol{\theta}$ is a column vector of n state variables, $\mathbf{F}$ is a column vector of n known functions, $\mathbf{u}$ is a control vector of M control variables, t is the independent variable time.

Assume that nominal control programs, $\mathbf{u}^N(t)$, and associates state variables $\boldsymbol{\theta}^N(t)$ are precalculated to produce a nominal path starting from a given initial point $\boldsymbol{\theta}(t_0)$ to a terminal point $\boldsymbol{\theta}(t_f)$, and t_f is the terminal time.

The implicit guidance system problem is that for small variations in the state from the nominal state, it is desired to find a control program which generates an optimum path, neighboring to the nominal path, from a given initial point to the terminal point with minimum control energy consumption.

The trajectories neighboring to the nominal path may be represented by the linearized version of Eq. (25) as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{m}(t) \quad (26)$$

where $\mathbf{x}(t)$ and $\mathbf{m}(t)$ are defined as

$$\mathbf{x}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}^N(t) \quad (27)$$

$$\mathbf{m}(t) = \mathbf{u}(t) - \mathbf{u}^N(t) \quad (28)$$

$\mathbf{A}(t)$ is a $n \times n$ matrix whose components are

$$a_{ij} = \left.\frac{\partial F_i}{\partial\theta_j}\right|_{\substack{\theta(t)=\theta^N(t)\\ \mathbf{u}(t)=\mathbf{u}^N(t)}} \qquad i,j = 1,2,\ldots,n \quad (29)$$

$\mathbf{B}(t)$ is a $n \times M$ matrix whose components are

$$b_{ij} = \left.\frac{\partial F_i}{\partial u_j}\right|_{\substack{\theta(t)=\theta^N(t)\\ \mathbf{u}(t)=\mathbf{u}^N(t)}} \qquad \begin{matrix} i = 1,2,\ldots,n \\ i = 1.1,2,\ldots,n \end{matrix} \quad (30)$$

If Eq. (26) is valid, the dynamic programing formulation of Section II, can be applied to find a feedback guidance law, which is linear to the state vector $\mathbf{x}(t)$, using a quadratic error measurement function h. The error measure is chosen as

$$h = \mathbf{m}^T(\sigma)\mathbf{m}(\sigma) + W\mathbf{x}^T(\sigma)\mathbf{x}(\sigma) \quad (31)$$

to minimize the control energy consumption and keep the trajectory close to the nominal path. Where W is a weighting factor which must be adjusted to keep the state vector $\mathbf{x}(t)$ small such that the linear Eq. (26) is valid. (The superscipt T denotes transpose.) For terminal guidance problem, the terminal constraints are

$$\mathbf{x}(t_f)\begin{cases} = \mathbf{0} \text{ for fixed-time guidance} \\ \neq \mathbf{0} \text{ for variable-time guidance} \end{cases} \quad (32)$$

and the number of constraints may be considered as $L = n$. Then the constraint functions I_i become

$$I_i = \dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{M} b_{ij}m_j \qquad i = 1,2,\ldots,n \quad (33)$$

Substituting Eqs. (31) and (33) into Eq. (22) gives the condition for minimum error function as (written in matrix form)

$$\underset{\mathbf{m}(\mu)\ \varepsilon\ \mathfrak{M}(\mu)}{\mathrm{Min}} \{\mathbf{m}^T(\mu)\mathbf{m}(\mu) + W\mathbf{x}^T(\mu)\mathbf{x}(\mu) + 2\lambda^T[\mathbf{A}(\mu)\mathbf{x}(\mu) + \mathbf{B}(\mu)\mathbf{m}(\mu)] + \mathbf{E}_{cx}{}^T[\mathbf{A}(\mu)\mathbf{x}(\mu) + \mathbf{B}(\mu)\mathbf{m}(\mu)]\} = -\frac{\partial E_c[\mathbf{x}(\mu),\mu]}{\partial\mu} \quad (34)$$

where

$$\lambda^T = (\lambda,\lambda_2\ldots\lambda_n)$$

$$\mathbf{E}_{cx}{}^T = [(\partial E_c/\partial x_1)(\partial E_c/\partial x_2)\ldots(\partial E_c/\partial x_n)]$$

Since in this case the minimum-error function E_c is in quadratic form of state variables, E_c is assumed to be as follows:

$$E_c[\mathbf{x}(\mu),\mu] = p(\mu) - 2\mathbf{Q}^T(\mu)\mathbf{x}(\mu) + \mathbf{x}^T(\mu)\mathbf{R}(\mu)\mathbf{x}(\mu) \quad (35)$$

where $p(\mu)$, $\mathbf{Q}(\mu)$, and $\mathbf{R}(\mu)$ are to be determined, and $\mathbf{Q}(\mu)$ is column vector of n components and $\mathbf{R}(\mu)$ is a symmetry $n \times n$ matrix. Then, taking the partial derivative gives

$$\partial E_c[\mathbf{x}(\mu),\mu]/\partial\mu = \dot{p}(\mu) - 2\dot{\mathbf{Q}}^T(\mu)\mathbf{x}(\mu) + \mathbf{x}^T(\mu)\dot{R}(\mu)x(\mu) \quad (36)$$

$$\mathbf{E}_{cx} = -2\mathbf{Q}(\mu) + 2\mathbf{R}(\mu)\mathbf{x}(\mu) \quad (37)$$

If the allowable region of the control signal is not limited, then, from Eq. (34), the optimum control signal is defined by the variational process

$$(\partial/\partial\mathbf{m})[\mathbf{m}^T(\mu)\mathbf{m}(\mu) + 2\lambda^T\mathbf{B}(\mu)\mathbf{m}(\mu) + \mathbf{E}_{cx}{}^T\mathbf{B}(\mu)\mathbf{m}(\mu)] = 0 \quad (38)$$

which gives the optimum control signal $\mathbf{m}^*(\mu)$ as

$$\mathbf{m}^*(\mu) = -\mathbf{B}^T(\mu)\lambda - \tfrac{1}{2}\mathbf{B}^T(\mu)\mathbf{E}_{cx} \quad (39)$$

substituting Eq. (37) into Eq. (39) yields

$$m^*(\mu) = -\mathbf{B}^T(\mu)\lambda + \mathbf{B}^T(\mu)\mathbf{Q}(\mu) - \mathbf{B}^T(\mu)\mathbf{R}(\mu)\mathbf{x}(\mu) \quad (40)$$

For the optimum control signal defined by Eq. (40), Eq. (34) becomes

$$\mathbf{m}^{*T}(\mu)\mathbf{m}^*(\mu) + W\mathbf{x}^T(\mu) + 2\lambda^T[\mathbf{A}(\mu)\mathbf{x}(\mu) + \mathbf{B}(\mu)\mathbf{m}^*(\mu)] + \mathbf{E}_{cx}{}^T[\mathbf{A}(\mu)\mathbf{x}(\mu) + \mathbf{B}(\mu)\mathbf{m}^*(\mu)] + \partial E_c[\mathbf{x}(\mu),\mu]/\partial\mu = 0 \quad (41)$$

Substituting Eqs. (36, 37, and 40) into Eq. (41) yields

$$[-\lambda^T\mathbf{B} + \mathbf{Q}^T\mathbf{B} - \mathbf{x}^T\mathbf{R}\mathbf{B}][-\mathbf{B}^T\lambda + \mathbf{B}^T\mathbf{Q} - \mathbf{B}^T\mathbf{R}\mathbf{x}] + W\mathbf{x}^T\mathbf{x} + 2\lambda^T\mathbf{A}\mathbf{x} + 2\lambda^T\mathbf{B}[-\mathbf{B}^T\lambda + \mathbf{B}^T\mathbf{Q} - \mathbf{B}^T\mathbf{R}\mathbf{x}] + [-2\mathbf{Q}^T + 2\mathbf{x}^T\mathbf{R}][\mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{B}^T\lambda + \mathbf{B}^T\mathbf{Q} - \mathbf{B}^T\mathbf{R}\mathbf{x})] + \dot{p} - 2\dot{\mathbf{Q}}^T\mathbf{x} + \mathbf{x}^T\dot{\mathbf{R}}\mathbf{x} = 0 \quad (42)$$

Note, the time functional symbol is dropped in the last expression for simplicity.

After some matrix manipulation, Eq. (42) can be written in the following form:

$$[\dot{p} - \lambda^T\mathbf{B}\mathbf{B}^T\lambda + 2\lambda^T\mathbf{B}\mathbf{B}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{B}\mathbf{B}^T\mathbf{Q}] + [-2\dot{\mathbf{Q}}^T + 2(\lambda^T - \mathbf{Q}^T)(\mathbf{A} - \mathbf{B}\mathbf{B}^T\mathbf{R})]\mathbf{x} + \mathbf{x}^T[\dot{\mathbf{R}} - \mathbf{R}\mathbf{B}\mathbf{B}^T\mathbf{R} + 2\mathbf{R}\mathbf{A} + W\mathbf{I}]\mathbf{x} = 0 \quad (43)$$

where $\mathbf{I}$ is a $n \times n$ identity matrix. This equation is satisfied for all $\mathbf{x}(\mu)$ only if the terms appearing in Eq. (43) vanish independently of $\mathbf{x}(\mu)$ so that

$$\dot{p}(\mu) = \lambda^T\mathbf{B}(\mu)\mathbf{B}^T(\mu)\lambda - 2\lambda^T\mathbf{B}(\mu)\mathbf{B}^T(\mu)\mathbf{Q}(\mu) + \mathbf{Q}^T(\mu)\mathbf{B}(\mu)\mathbf{B}^T(\mu)\mathbf{Q}(\mu) \quad (44)$$

$$\dot{\mathbf{Q}}(\mu) = [\mathbf{A}^T(\mu) - \mathbf{R}(\mu)\mathbf{B}(\mu)\mathbf{B}^T(\mu)][\lambda - \mathbf{Q}(\mu)] \quad (45)$$

$$\dot{\mathbf{R}}(\mu) = \mathbf{R}(\mu)\mathbf{B}(\mu)\mathbf{B}^T(\mu)\mathbf{R}(\mu) - 2\mathbf{R}(\mu)\mathbf{A}(\mu) - W\mathbf{I} \quad (46)$$

The boundary conditions are

$$p(t_f) = 0, \mathbf{Q}(t_f) = 0, \mathbf{R}(t_f) = 0 \quad (47)$$

It is seen that the feedback gain matrix specified by Eq. (46) for the optimal guidance law (40) is a set of nonlinear differential equations independent of the Lagrange multipliers λ. Hence, $\mathbf{R}(\mu)$ can be precalculated without concern of the terminal constraints. But the feedforward portion of the optimal control signal and $\mathbf{Q}(\mu)$ are linear functions of λ. The optimal control signal, $\mathbf{m}^*(\mu)$ cannot be obtained until the values of λ which satisfy the terminal constraint are known. If analytic solution of the state equations cannot be obtained, one method to compute the λ is: try different values of λ to calculate $\mathbf{Q}(\mu)$, $\mathbf{m}^*(\mu)$ and then solve the state equation for $\mathbf{x}(\mu)$ such that the two-point boundary conditions of the state variables are satisfied. This is a time-consuming procedure which prevents the effective use of Lagrange multiplier method for solving the constraint problems. In the next section, a direct computation method for the Lagrange Multipliers λ is developed.

IV. Computation of the Lagrange Multipliers

Let

$$\mathbf{V}(\mu) = \mathbf{Q}(\mu) - \lambda \quad (48)$$

then Eqs. (40) and (45) become

$$\mathbf{m}^*(\mu) = \mathbf{B}^T(\mu)\mathbf{V}(\mu) - \mathbf{B}^T(\mu)\mathbf{R}(\mu)\mathbf{x}(\mu) \quad (49)$$

and

$$\dot{\mathbf{V}}(\mu) = [\mathbf{R}(\mu)\mathbf{B}(\mu)\mathbf{B}^T(\mu) - \mathbf{A}^T(\mu)]\mathbf{V}(\mu) \quad (50)$$

$$\mathbf{V}(t_f) = -\lambda \quad (51)$$

Let

$$\mathbf{C}(\mu) = [\mathbf{R}(\mu)\mathbf{B}(\mu)\mathbf{B}^T(\mu) - \mathbf{A}^T(\mu)] \quad (52)$$

and, let $\phi(\mu,t)$ be the $n \times n$ transition matrix such that

$$\mathbf{V}(\mu) = \phi(\mu,t_0)\mathbf{V}(t_0) \quad (53)$$

From the properties of the state transition matrix,[7] it can be shown that

$$\phi(\mu,t) = \phi^{-1}(t,\mu) \quad (54)$$

$$d/d\mu[\phi(\mu,t_0)] = \mathbf{C}(\mu)\phi(\mu,t_0) \quad (55)$$

$$d/d\mu[\phi(t_0,\mu)] = -\phi(t_0,\mu)\mathbf{C}(\mu) \quad (56)$$

$$\phi(t_0,t_0) = I \quad (57)$$

Therefore, Eqs. (51, 53, and 54) give

$$\mathbf{V}(\mu) = -\phi(\mu,t_0)\phi(t_0,t_f)\lambda \quad (58)$$

Substitution Eq. (58) into Eq. (49), gives

$$\mathbf{m}^*(\mu) = -\mathbf{B}^T(\mu)\phi(\mu,t_0)\phi(t_0,t_f)\lambda - \mathbf{B}^T(\mu)\mathbf{R}(\mu)\mathbf{x}(\mu) \quad (59)$$

And substituting this $\mathbf{m}^*(\mu)$ for $m(\mu)$ into Eq. (26) gives

$$\dot{\mathbf{x}}(\mu) = [\mathbf{A}(\mu) - \mathbf{B}(\mu)\mathbf{B}^T(\mu)\mathbf{R}(\mu)]\mathbf{x}(\mu) - \mathbf{B}(\mu)\mathbf{B}^T(\mu) \times \phi(\mu,t_0)\phi(t_0,t_f)\lambda = -\mathbf{C}^T(\mu)\mathbf{x}(\mu) - \mathbf{B}(\mu)\mathbf{B}^T(\mu) \times \phi(\mu,t_0)\phi(t_0,t_f)\lambda \quad (60)$$

The complete solution of Eq. (60) is found as follows: premultiplying Eq. (60) by $\phi^T(\mu,t_0)$ gives

$$\boldsymbol{\phi}^T(\mu,t_0)\dot{\mathbf{x}}(\mu) = -\boldsymbol{\phi}^T(\mu,t_0)\mathbf{C}^T(\mu)x(\mu) - \boldsymbol{\phi}^T(\mu,t_0)\mathbf{B}(\mu)\mathbf{B}^T \times \boldsymbol{\phi}(\mu,t_0)\boldsymbol{\phi}(t_0,t_f)\boldsymbol{\lambda} \quad (61)$$

postmultiplying the transpose of Eq. (55) by $\mathbf{x}(\mu)$ gives

$$\dot{\boldsymbol{\phi}}^T(\mu,t_0)\mathbf{x}(\mu) = \boldsymbol{\phi}^T(\mu,t_0)\mathbf{C}^T(\mu)\mathbf{x}(\mu) \quad (62)$$

Adding Eq. (61) and Eq. (62) gives

$$d/d\mu[\boldsymbol{\phi}^T(\mu,t_0)\mathbf{x}(\mu)] = -\boldsymbol{\phi}^T(\mu,t_0)\mathbf{B}(\mu)\mathbf{B}^T(\mu) \times \boldsymbol{\phi}(\mu,t_0)\boldsymbol{\phi}(t_0,t_f)\boldsymbol{\lambda} \quad (63)$$

Integrating Eq. (63) from t_0 to μ gives

$$\mathbf{x}(\mu) = [\boldsymbol{\phi}^T(\mu,t_0)]^{-1}\mathbf{x}(t_0) - [\boldsymbol{\phi}^T(\mu,t_0)]^{-1}\int_{t_0}^{\mu}\boldsymbol{\phi}^T(\sigma,t_0) \times \mathbf{B}(\sigma)\mathbf{B}^T(\sigma)\boldsymbol{\phi}(\sigma,t_0)\boldsymbol{\phi}(t_0,t_f)\boldsymbol{\lambda}d\sigma \quad (64)$$

or

$$\mathbf{x}(\mu) = \boldsymbol{\phi}^T(t_0,\mu)\mathbf{x}(t_0) - [\boldsymbol{\phi}^T(t_0,\mu)\int_{t_0}^{\mu}\boldsymbol{\phi}^T(\sigma,t_0)\mathbf{B}(\sigma)\mathbf{B}^T(\sigma) \times \boldsymbol{\phi}(\sigma,t_0)d\sigma]\boldsymbol{\phi}(t_0,t_f)\boldsymbol{\lambda} \quad (65)$$

Let

$$\mathbf{D}(t_0,t_f) = \boldsymbol{\phi}^T(t_0,t_f)\int_{t_0}^{t_f}\boldsymbol{\phi}^T(\sigma,t_0)\mathbf{B}(\sigma)\mathbf{B}^T(\sigma) \times \boldsymbol{\phi}(\sigma,t_0)d\sigma \quad (66)$$

then, Eq. (65) and Eq. (60) give

$$\mathbf{x}(t_f) = \boldsymbol{\phi}^T(t_0,t_f)\mathbf{x}(t_0) - \mathbf{D}(t_0,t_f)\boldsymbol{\phi}(t_0,t_f)\boldsymbol{\lambda} \quad (67)$$

Hence,

$$\boldsymbol{\lambda} = \boldsymbol{\phi}(t_f,t_0)\mathbf{D}^{-1}(t_0,t_f)[\boldsymbol{\phi}^T(t_0,t_f)\mathbf{x}(t_0) - \mathbf{x}(t_f)] \quad (68)$$

Finally, substituting Eq. (68) into Eq. (59) gives the optimal terminal guidance law as

$$\mathbf{m}^*(\mu) = -\mathbf{B}^T(\mu)\boldsymbol{\phi}(\mu,t_0)\mathbf{D}^{-1}(t_0,t_f)[\boldsymbol{\phi}^T(t_0,t_f)\mathbf{x}(t_0) - \mathbf{x}(t_f)] - \mathbf{B}^T(\mu)\mathbf{R}(\mu)\mathbf{x}(\mu) \quad (69)$$

Then Eq. (28) gives the optimal control vector $\mathbf{u}^*(\mu)$ as

$$\mathbf{u}^*(\mu) = \mathbf{u}^N(\mu) + \mathbf{m}^*(\mu) \quad (70)$$

From Eq. (69), it is seen that the feedback gains and all the other matrices appearing in the control equation are given or precalculated except the state vector $\mathbf{x}(\mu)$. Hence, the control system can be easily synthesized with a relatively simple onboard guidance computer. In view of Eq. (69), this control scheme is good for any $\mathbf{x}(t_0)$ and $\mathbf{x}(t_f)$, and hence, good for solving two-point-boundary-value problems and optimization and control of nonlinear systems.

V. Conclusion

It has been shown that dynamic programing formulation with integral-type constraints can be applied easily to terminal guidance problems. It is attractive that the computation is straightforward, and the solution is optimal with zero terminal errors for linear systems. Furthermore, it can be effectively applied to synthesize the guidance system for aerospace vehicle guidance using simple guidance computers.

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